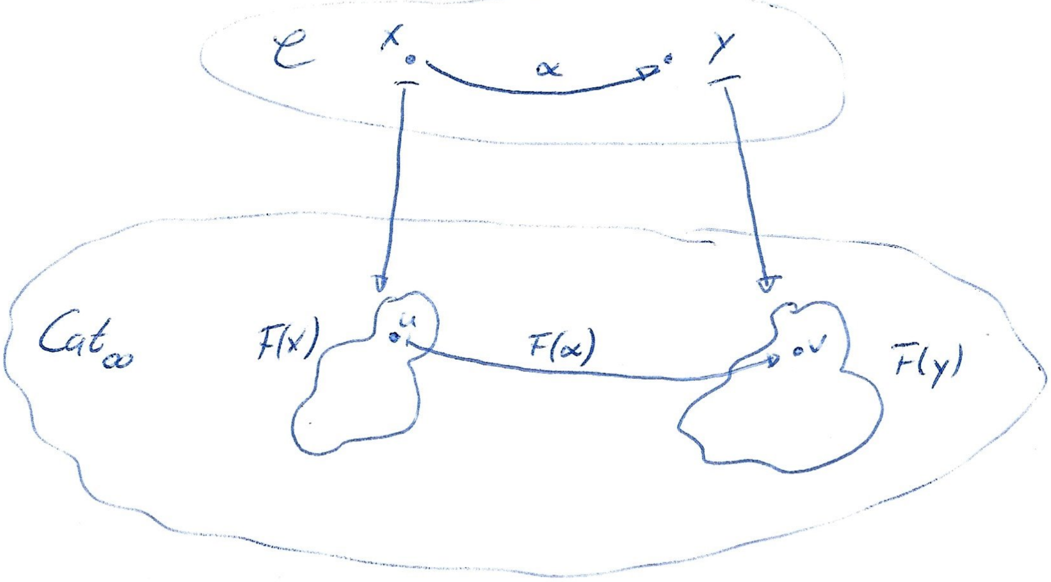
Lurie's straightening equivalence

Goals:
- construct a functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{A}n$ for $\mathcal{C} \in \mathcal{C}at_\infty$
  ↳ via straightening/unstraightening.
- prove the ∞-Yoneda Lemma.

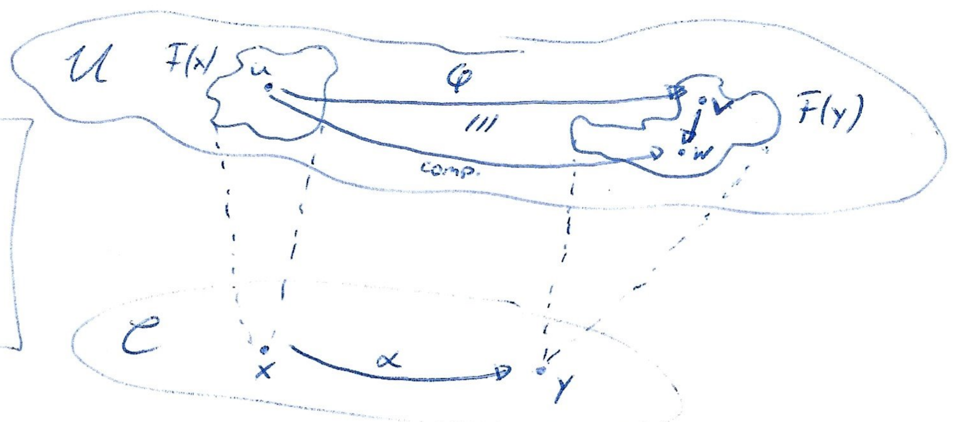
Motivation: How do we construct functors $F: \mathcal{C} \to \mathcal{A}n$ or $F: \mathcal{C} \to \mathcal{C}at_\infty$?

Let $x, y \in \mathcal{C}$, $\alpha: x \to y$ morphism. Then, for a functor $F: \mathcal{C} \to \mathcal{C}at_\infty$,
$F(x), F(y)$ are quasi-categories and $F(\alpha): F(x) \to F(y)$.

$\Rightarrow \forall u \in F(x) \ \exists v$ s.t. $v \simeq F(\alpha)(u) \in F(y)$.



Flip:



$\Delta^2 \to \mathcal{U}$

Think: $p: \mathcal{U} \to \mathcal{C}$ fibration.

$F(x), F(y)$ as fibres over x, y.

$\mathcal{U}$ union of fibres.

But, we need to encode "F varying functorially in x" into $\mathcal{U}$:

For every $u \in F(x)$ and some $v \simeq F(\alpha)(u)$ in the image connect them by a 1-simplex. Keep adding simplices until we obtain a quasi-category.

$\boxed{\text{Example}}$ (i) R ring, LMod_R left modules over R. R' another ring. with $R \to R$ ringhom.

<u>idea:</u> There should be a functor $\overset{R}{\cancel{\mathrm{LMod}_R}} \to \mathrm{LMod}_R$ with

Base change $R' \otimes_R - : \mathrm{LMod}_R \to \mathrm{LMod}_{R'}$!

<u>However:</u> For R'' ring with $R' \to R''$ ringhom. we only have

$$R'' \otimes_{R'} (R' \otimes_R -) \simeq R'' \otimes_R - \quad \text{(not equality!)}$$

<u>Hence:</u> Any functor $\mathrm{LMod}_{(-)}$ needs to encode such equivalences.

$$\Rightarrow \quad \mathrm{LMod}_{(-)} : N(\mathrm{Ring}) \to \mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty.$$

hard to construct.

(ii) Lets study the fibration! Let LMod be cat. with pairs $(\overset{R\text{-module}}{M}, \overset{\text{Ring}}{R})$ and morphisms $(R \xrightarrow{\alpha} R', R' \otimes_R M \xrightarrow{\varphi} M')$. Then, proj : LMod $\to$ Ring.

induces

$$p : N(\mathrm{LMod}) \to N(\mathrm{Ring}) \text{ with fibres}$$

$$p^{-1}\{R\} \simeq N(\{R\} \overset{\text{pullback}}{\times_{\mathrm{Ring}}} \mathrm{LMod}) \simeq N(\mathrm{LMod}_R).$$

On morphisms (for ringhom $R \to R'$)

$$\begin{tikzcd}
(R, M) \arrow[r, "\text{any}"] \arrow[d, "\text{canonical}"'] & (R', M') \\
(R', R' \otimes_R M) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd} \quad \rightsquigarrow \quad \sigma : \Delta^2 \to N(\mathrm{LMod}).$$

Conclusion: Functors $\mathcal{C} \to \mathcal{C}at_\infty$ correspond to certain fibrations $p: \mathcal{U} \to \mathcal{C}$, s.t.

$F(x) \rightleftarrows$ fibres $p^{-1}(\{x\})$ and morphisms are related by simplexes.

$\Rightarrow$ Definition: Let $p: \mathcal{U} \to \mathcal{C}$ inner fibration of quasi-cat.

(a) A morphism $\varphi: u \to v$ in $\mathcal{U}$ is called p-cocartesian if, for every $n \geq 2$, every lifting

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

in which $\Delta^{\{0,1\}} \subseteq \Lambda^n_0$ is sent to φ, has a solution.

(b) We call p a cocartesian fibration if every lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d, "p"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

has a solution in which Δ^1 is sent to a p-cocartesian morphism.

(c) The dual notions p-cartesian and cartesian fibration use $\cancel{\Delta^{\{n-1,n\}} \to \Lambda^n_n \subseteq \Delta^n}$ and $\{1\} \to \Delta^1$ instead.

$\Delta^{\{n-1,n\}} \subseteq \Lambda^n_n \to \Delta^n$

Lemma 5.3: ("Left fibrations are cocartesian fibrations whose fibres are animae")

p cocartesian fibration, $p: \mathcal{U} \to \mathcal{C}$, then TFAE:

(a) Every morphism in $\mathcal{U}$ is p-cocartesian

(b) p is a left fibration.

(c) All fibres $p^{-1}\{x\}$ for $x \in \mathcal{C}$ are animae.

explains the special case $F: \mathcal{C} \to \mathrm{An}$.

proof: (a) $\Leftrightarrow$ (b): Clear.

(b) $\Rightarrow$ (c) last time.

(c) $\Rightarrow$ (a): Idea: We need a solution to

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\sigma"] \arrow[d] & \mathcal{U} \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "?"] & \mathcal{C} \end{tikzcd}$$

s.t. $\sigma|_{\Delta^{\{0,1\}}}$ is φ.

construct and solve

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "d_1"] \arrow[d] & \Lambda^{n+1}_1 \arrow[r, "\bar{\sigma}"] \arrow[d] & \mathcal{U} \arrow[d, "p"] \\ \Delta^n \arrow[r, "d_1"] \arrow[ur, phantom, "\cong"] & \Delta^{n+1} \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

here: $\Delta^{\{0,1\}}$ is image of φ under $\bar{\sigma}$.

The solution is obtained as p is an inner fibration.

$\bar{\sigma}$ is constructed utilizing Joyal's lifting theorem and $p^{-1}\{x\}$ anima.

□

Theorem: (Straightening/Unstraightening) (Lurie, 2009)

Let $\mathcal{C}$ be a quasi-category.

(a) Let $\mathrm{Cocart}(\mathcal{C}) \subseteq \mathrm{Cat}_{\infty/\mathcal{C}}$ be the (non-full) sub-quasi-category spanned by fibrations over $\mathcal{C}$ and those morphisms (in $\mathrm{Cat}_{\infty/\mathcal{C}}$) that preserve cocartisian morphisms.

Then there are inverse equivalences of quasi-categories

$$\underset{\text{"straightening"}}{St^{cocart}} : \mathrm{Cocart}(\mathcal{C}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} F(\mathcal{C}, \mathrm{Cat}_\infty) : \underset{\text{"unstraightening"}}{Un^{cocart}}.$$

For a cocartesian fibration, $p: \mathcal{U} \to \mathcal{C}$, the value of the functor $St^{cocart}(p): \mathcal{C} \to \mathrm{Cat}_\infty$, $x \mapsto p^{-1}\{x\}$ at $x \in \mathcal{C}$ is $p^{-1}\{x\}$.

Furthermore, for any functor $F: \mathcal{C} \to \mathcal{D}$ the unstraightening equivalence identifies the precomposition functor with pullback

$$- \circ F : F(\mathcal{D}, \mathrm{Cat}_\infty) \to F(\mathcal{C}, \mathrm{Cat}_\infty) \longmapsto F^* : \mathrm{Cocart}(\mathcal{D}) \to \mathrm{Cocart}(\mathcal{C}).$$

(b) For $\mathrm{Left}(\mathcal{C})$, the full-sub-quasi-category of left fibrations, the equivalence restricts to

$$St^{left} : \mathrm{Left}(\mathcal{C}) \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} F(\mathcal{C}, \mathcal{A}n) : Un^{left}.$$

Remark: There are dual equivalences

$\mathrm{Cart}(\mathcal{C}) \simeq F(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$ and $\mathrm{Right}(\mathcal{C}) \simeq F(\mathcal{C}^{op}, \mathcal{A}n)$. (black box) ▨

Remark: On the naming conventions:

- straightening: takes the collection of fibres and ~~[illegible]~~ encodes their data into a single (straight!) arrow
- unstraightening: takes a single arrow and "decomposes" it into its individual fibres.

• Proof in J. Lurie, "Higher topos theory", 2009

Examples: ⓪ $\mathcal{D}$ quasi-category. Functor $\mathcal{D} \to *$ is a cocartesian fibration. (cocart. morphisms are equivalences in $\mathcal{D}$).

Further, $\mathcal{D} \times \mathcal{C} \xrightarrow{\mathrm{proj}_2} \mathcal{C}$ is a cocartesian fibration $\forall\, \mathcal{C} \in \mathrm{Cat}_\infty$.

$$\left(\text{pullback } \begin{array}{ccc} \mathcal{D}\times\mathcal{C} & \to & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D} & \to & * \end{array}\right)$$

proj_2-cocartesian morphisms are equivalences in the $\mathcal{D}$-component.

$$\Rightarrow \quad St^{\mathrm{cocart}}(\mathrm{proj}_2) = \mathrm{const}\, \mathcal{D} : \mathcal{C} \to \mathrm{Cat}_\infty .$$

① The functor $p: N(\mathrm{Ring}) \to N(\mathrm{LMod})$ is a cocartesian fibration.

Lifts of $R \to R'$ are given by $(R, M) \to (R', R' \otimes_R M)$.

We can define

$$\mathrm{LMod}_{(-)} := St^{\mathrm{cocart}}(p) : \mathrm{Ring} \to \mathrm{Cat}_\infty .$$

The fibres are $N(\mathrm{LMod}_R)$ and $\mathrm{LMod}_{(-)}$ factors through $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$.

Remark: More generally, fibred category: functor $p: \mathcal{U} \to \mathcal{E}$ (ordinary categories) s.t. $N(p): N(\mathcal{U}) \to N(\mathcal{E})$ cartesian fibration. Then, $St^{\mathrm{cart}}: N(\mathcal{E})^{op} \to \mathrm{Cat}_\infty$ always factor through $\mathrm{Cat}^{(2)} \subseteq \mathrm{Cat}_\infty$.

(2) <u>Target projection</u> $t : \underset{\mathrm{Fun}(\Delta^1, \mathcal{C})}{\overset{\shortparallel}{\mathrm{Ar}(\mathcal{C})}} \to \mathcal{C}$ is a cocartesian fibration. We obtain / can define the functor

$$\mathcal{C}_{/-} := \mathrm{St}^{\mathrm{Cocart}}(t) : \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$$
$$x \longmapsto \mathcal{C}_{/x} \quad \text{(slice/comma category)}$$

<u>Fact:</u> A morphism φ in $\mathrm{Ar}(\mathcal{C})$ is t-cocartesian iff

$u \to v$ is an equivalence.

⌈ <u>Here note:</u>

$$\begin{tikzcd} u \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\equiv"] & v \arrow[d, "\beta"] \\ u' \arrow[r] & v' \end{tikzcd}$$

$\varphi : \alpha \to \beta$. ⌋

(3) Let M be a monoid. Then

$$F(BM, \mathrm{Cat}_\infty) \simeq \mathrm{Cocart}(BM)$$

Since BM has only one object $*$, every functor $F : BM \to \mathrm{Cat}_\infty$ cosists of the image $F(*)$ and a map (of monoids)

$$M \longrightarrow F(\mathcal{C}, \mathcal{C}).$$

Hence

$$\mathrm{Cocart}(BM) \simeq \{\infty\text{-category } \mathcal{C} \text{ with a homotopy coherent left monodidl action of } M\}.$$

(4) For every $x \in \mathcal{C}$, $t : \mathcal{C}_{x/} \to \mathcal{C}$ is a left fibration with fibre $\mathrm{Hom}_{\mathcal{C}}(x, y)$, over $y \in \mathcal{C}$ we can define ~~the~~ a part of our hom-functor

$$\mathrm{Hom}_{\mathcal{C}}(x, -) := \mathrm{St}^{\mathrm{Cocart}}(t) : \mathcal{C} \longrightarrow \mathrm{An}.$$

Similarly, $s : \mathcal{C}_{/y} \to \mathcal{C}$ is a right fibration and $\mathrm{Hom}_{\mathcal{C}}(-, y) := \mathrm{St}^{\mathrm{Cart}}(s) : \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$.

Straightening / Unstraightening on morphisms. (Haugseng)

$p: \mathcal{U} \to \mathcal{C}$ cocartesian fibration, $F := st^{\text{Cocart}}(p): \mathcal{C} \to \mathrm{Cat}_\infty$.

Our goal is to describe $F(\alpha): F(x) \to F(y)$ for some morphism $\alpha: x \to y$ in $\mathcal{C}$.

Let $Ar^{\text{Cocart}}(\mathcal{U}) \subseteq Ar(\mathcal{U})$ denote the full-quasi-subcategory spanned by p-cocartesian morphisms.

Fact: $Ar^{\text{Cocart}}(\mathcal{U}) \to Ar(\mathcal{C}) \underset{\text{pullback}}{\times_{s,\mathcal{C}}} \mathcal{U}$ is a trivial fibration.

Idea: given $\alpha: x \to y$ in $\mathcal{C}$ and $u \in p^{-1}\{x\}$, then the lift of α to $\varphi: u \to v$ exists and is unique (up to homotopy)

Now for $F(\alpha): p^{-1}\{x\} \to p^{-1}\{y\}$:

- Pull back $Ar^{\text{Cocart}}(\mathcal{U}) \to Ar(\mathcal{C}) \times_{s,\mathcal{C}} \mathcal{U}$ along $\{\alpha\} \to Ar(\mathcal{C})$

$\rightsquigarrow$ trivial fibration

$$\{\alpha\} \times_{Ar(\mathcal{C})} Ar(\mathcal{U}) \to \{\alpha\} \times_{s,\mathcal{C}} \mathcal{U} \cong \{x\} \times_{\mathcal{C}} \mathcal{U} \cong p^{-1}\{x\}.$$

Each trivial fibration admits a section.

- compose (the chosen) section with the target projections $t: Ar(\mathcal{U}) \to \mathcal{U}$ and $t: Ar(\mathcal{C}) \to \mathcal{C}$

$$\rightsquigarrow \quad F: p^{-1}\{x\} \xrightarrow{\text{section}} \{\alpha\} \times_{Ar(\mathcal{C})} Ar(\mathcal{U}) \xrightarrow{t} \{y\} \times_{\mathcal{C}} \mathcal{U} \cong p^{-1}\{y\}.$$

Rest: [Lan 21, §3.3].

M. Land, "Introduction to infinity categories", 2021.

Interlude: Homotopy pullbacks

"Definition":

(a) Let $J := \{\bullet \underset{\alpha}{\rightleftarrows} \bullet\}$ the cat. with two objects and a pair of mutually inverse morphisms. An Isofibration, or categorical fibration, is an inner fibration with lifting against $\{0\} \to N(J)$. (i.e. we can lift equivalences)

(b) A diagramm of Kan complexes is a homotopy pullback if its homotopy equivalent to an (honest) pullback of Kan complexes in which one leg is a Kan fibration.

(c) A diagramm of quasi-categories is a homotopy-pullback if it is equivalent to an honest) pullback of quasi-categories in which one leg is an isofibration.

Lemma: $\mathcal{C} \in Cat_\infty$, $\alpha : x \to y$, $\alpha' : x' \to y'$ morph. in $\mathcal{C}$. Then $\exists$ homotopy pullback of animae

$$\begin{tikzcd}
\mathrm{Hom}_{Ar(\mathcal{C})}((\alpha : x \to y), (\alpha' : x' \to y')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\, h"] & \mathrm{Hom}_{\mathcal{C}}(y, y') \arrow[d, "\alpha^* \text{ pre-comp.}"] \\
\mathrm{Hom}_{\mathcal{C}}(x, x') \arrow[r, "\alpha'_*"', "\text{post-comp.}"] & \mathrm{Hom}(x, y')
\end{tikzcd}$$

$\square$

<u>Corollary</u>: $\mathcal{C} \in \mathrm{Cat}_\infty$, $y \in \mathcal{C}$, $\alpha : x \to y$, $\alpha' : x' \to y$ morphisms in $\mathcal{C}$.

Then $\exists$ homotopy pullback of animae

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}/y}((\alpha : x \to y), (\alpha' : x' \to y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & \{\alpha\} \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(x, x') \arrow[r, "\alpha'_*"] & \mathrm{Hom}_{\mathcal{C}}(x, y)
\end{tikzcd}$$

- Proof again J. Lurie, "Higher topos theory", 2009.

<u>Lemma</u>: $p : \mathcal{U} \to \mathcal{C}$ inner fibration of quasi-categories.

Then $\varphi : u \to v$ in $\mathcal{U}$ is p-cocartesian if and only if

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{U}}(v, w) \arrow[r, "\varphi^*"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner h"] & \mathrm{Hom}_{\mathcal{U}}(u, w) \arrow[d, "p"] \\
\mathrm{Hom}_{\mathcal{C}}(p(v), p(w)) \arrow[r, "p(\varphi)^*"] & \mathrm{Hom}_{\mathcal{C}}(p(u), p(w))
\end{tikzcd}$$

is a homotopy pullback of animae for all $w \in \mathcal{U}$.

<u>Remark</u>: This can be used to define p-cocartesian morphisms, independently of model.

Further p cocart. fib. $\Leftrightarrow$

$$\begin{tikzcd}
\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathcal{C}) \arrow[d, "s"] \\
\mathcal{U} \arrow[r, "p"] & \mathcal{C}
\end{tikzcd}$$

is a pullback in Cat_∞.

The Yoneda Lemma

Theorem: (Quasi-categorical Yoneda Lemma)
Let $\mathcal{C}$ be a quasi category, $x \in \mathcal{C}$ an object, and $E: \mathcal{C} \to An$ a functor. Then evaluation at id_x induces an equivalence of animae

$$ev_{id_x}: Hom_{F(\mathcal{C}, An)}(Hom_{\mathcal{C}}(x,-), E) \xrightarrow{\simeq} E(x).$$

Dually, for $E: \mathcal{C}^{op} \to An$

$$ev_{id_y}: Hom_{F(\mathcal{C}^{op}, An)}(Hom_{\mathcal{C}}(-,y), E) \xrightarrow{\simeq} E(\cancel{x}).$$

Lemma: $\mathcal{C} \in Cat_\infty$, $x \in \mathcal{C}$. Then for every left fibration $X \to \mathcal{C}$, the natural map $F(\mathcal{C}_{/x}, X) \xrightarrow{\simeq} F(*, X) \times_{F(x, \mathcal{C})} F(\mathcal{C}_{x/}, \mathcal{C})$ is an equivalence of categories.

Proof: Notes. ☒

Proof of Yoneda: Let $p: \mathcal{U} \to \mathcal{C}$ be the unstraightening of E. Then

un/straightening

$$Hom_{F(\mathcal{C}, An)}(Hom_{\mathcal{C}}(x,-), E) \simeq Hom_{Left(\mathcal{C})}(t: \mathcal{C}_{x/} \to \mathcal{C}, p: \mathcal{U} \to \mathcal{C})$$

$$\simeq Hom_{Cat_\infty}(t: \mathcal{C}_{x/} \to \mathcal{C}, p\,\mathcal{U} \to \mathcal{C})$$

Left(C) full sub cat.

Thus we can study $\mathrm{Hom}_{Cat_\infty}((t:\mathcal{C}_{x/}\to\mathcal{C}),(p:\mathcal{U}\to\mathcal{C}))$.
However, this is a homotopy pullback (collorary)

$$\begin{tikzcd}
\mathrm{Hom}_{Cat_\infty}((t:\mathcal{C}_{x/}\to\mathcal{C}),(p:\mathcal{U}\to\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & * \arrow[d] & * \arrow[d, mapsto] \\
\mathrm{Hom}_{Cat_\infty}(\mathcal{C}_{x/},\mathcal{U}) \arrow[r] & \mathrm{Hom}_{Cat_\infty}(\mathcal{C}_{x/},\mathcal{C}). & t
\end{tikzcd}$$

Further, recall $\mathrm{Hom}_{Cat_\infty}(-,-)\simeq \mathrm{core}\,F(-,-)$ and we have homotopy pullbacks

$$\begin{tikzcd}
F(\mathcal{C}_{x/},\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & F(\mathcal{C}_{x/},\mathcal{C}) \arrow[d] \\
\mathcal{U} \arrow[r, "p"'] & \mathcal{C}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{core}\,F(\mathcal{C}_{x/},\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & \mathrm{core}\,F(\mathcal{C}_{x/},\mathcal{C}) \arrow[d] \\
\mathrm{core}(\mathcal{U}) \arrow[r] & \mathrm{core}(\mathcal{C})
\end{tikzcd}$$

($p:\mathcal{U}\to\mathcal{C}$ is an isofibration as any left fibration has lifting against $\{0\}\to N(J)$).

+ prev. Lemma
$F(\mathcal{C}_{x/},\mathcal{U})$ is an ordinary pullback

(core is right adjoint to the inclusion $An\hookrightarrow Cat_\infty$ [later] or directly ~~on~~ involving Joyal)

Core turns isofibrations into Kan fibration.

$\Rightarrow$

$$\begin{tikzcd}
\mathrm{Hom}_{Cat_\infty}((t:\mathcal{C}_{x/}\to\mathcal{C}),(p:\mathcal{U}\to\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & * \arrow[d] & * \arrow[d, mapsto] \\
\mathrm{core}(\mathcal{U}) \arrow[r, "Kan"'] & \mathrm{core}(\mathcal{C}) \ni x & x
\end{tikzcd}$$

For Kan fibrations the ordinary and homotopy pullbacks agree. Hence

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}((t:\mathcal{C}_{x/}\to\mathcal{C}),(p:\mathcal{U}:\mathcal{C})) \simeq \mathrm{core}(\mathcal{U}) \times_{\mathrm{core}(\mathcal{C})} \{x\}$$

$$\cong \mathrm{core}(\mathcal{U}\times_{\mathcal{C}}\{x\}) \cong \mathcal{U}\times_{\mathcal{C}}\{x\} = p^{-1}\{x\}$$

$$\simeq F(x).$$

[Remark: The yoneda Lemma is functorial in x and E.]

$\square$

Construction/Definition

$\mathcal{C}_{\angle}: \mathcal{C} \to \mathrm{Cat}_\infty$ functor. Every $x \in \mathcal{C}$ yields $\mathcal{C}_{/x} \to \mathcal{C}$ natural functor.

$\Rightarrow$ There should be a functor $\mathcal{C}_{/-}: \mathcal{C} \to \mathrm{Cat}_\infty/\mathcal{C}$.

Fact: For all quasi categories $\mathcal{D}$, $y \in \mathcal{D}$: $F(\mathcal{C},\mathcal{D}_{/y}) \simeq F(\mathcal{C},\mathcal{D})/_{\mathrm{const}\, y}$

Further ~~Thus~~: $F(\mathcal{C},\mathrm{Cat}_\infty)/_{\mathrm{const}\,\mathcal{C}} \underset{\text{un/straightening}}{\simeq} \mathrm{Cocart}(\mathcal{C})/_{(pr_2:\mathcal{C}\times\mathcal{C}\to\mathcal{C})}$

$\Rightarrow$ In order to lift we just observe that

$$\begin{tikzcd} Ar(\mathcal{C}) \arrow[r, "{(s,t)}"] \arrow[d, "t"'] & \mathcal{C}\times\mathcal{C} \arrow[dl, "pr_2"] \\ \mathcal{C} \arrow[ur, phantom, "\cong"] & \end{tikzcd}$$

is a morphism of cocartesian fibrations over $\mathcal{C}$.

(recall $\mathcal{C}_{/-} = St^{\mathrm{Cocart}}(t)$)

We saw before that $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration.

Consequently, $\mathcal{C}_{/-}: \mathcal{C} \to \mathrm{Cat}_\infty/\mathcal{C}$ factors through the full sub quasi cat. $\mathrm{Right}(\mathcal{C}) \subseteq \mathrm{Cat}_\infty/\mathcal{C}$.

Since $\mathrm{Right}(\mathcal{C}) \simeq \mathcal{F}(\mathcal{C}^{op}, \mathrm{An})$ we obtain a functor

$$\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \mathrm{Right}(\mathcal{C}) \xrightarrow{St^{right}} \mathcal{F}(\mathcal{C}^{op}, \mathrm{An}) =: \mathrm{PSh}(\mathcal{C})$$

"Yoneda embedding" $\left(\Rightarrow \mathcal{Y}_{\mathcal{C}} \in \mathcal{F}(\mathcal{C}, \mathcal{F}(\mathcal{C}^{op}, \mathrm{An})) \right)$ "∞-category of Presheaves on $\mathcal{C}$"

Finally, let $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{An}$ be the image of $\mathcal{Y}_{\mathcal{C}}$ under "currying". $\mathcal{F}(\mathcal{C}, \mathcal{F}(\mathcal{C}^{op}, \mathrm{An})) \simeq \mathcal{F}(\mathcal{C}^{op} \times \mathcal{C}, \mathrm{An})$

Definition: The twisted-arrow quasi-category $(s,t) : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$ is the unstraightening of $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{An}$.

objects: arrows $(\alpha : x \to y)$

morphisms: $\varphi : (\alpha : x \to y) \to (\beta : x' \to y')$ twisted.

are comm. squares
$$\begin{tikzcd} x & x' \arrow[l] \arrow[d] \\ y \arrow[u, leftarrow] \arrow[r] & y' \end{tikzcd}$$

Lemma: For $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{An}$, $x, y \in \mathcal{C}$ the restrictions

$$\left(\mathrm{Hom}_{\mathcal{C}}|_{\{x\} \times \mathcal{C}} : \mathcal{C} \to \mathrm{An} \right) = \mathrm{Hom}_{\mathcal{C}}(x, -)$$

and

$$\left(\mathrm{Hom}_{\mathcal{C}}|_{\mathcal{C} \times \{y\}} : \mathcal{C}^{op} \to \mathrm{An} \right) = \mathrm{Hom}_{\mathcal{C}}(-, y).$$

Corollary: $\mathcal{Y}_{\mathcal{C}}$ is fully faithful for every quasi-category $\mathcal{C}$.